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A note on some nonlinear water-wave experiments and the comparison of data with theory

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The problem of the instability of a uniform, nonlinear, deep-water wave train to infinitesimal long-wave perturbations, first studied by Benjamin & Feir (1967) and Benjamin (1967), is re-examined. It is found that the apparent discrepancy between the experimental and theoretical growth rates of the instability is associated with the experimental generation of waves which do not have the Stokes wave profiles assumed in the theory. Experimental and theoretical results relating the initial wave steepness and the most unstable long-wave perturbation are used to obtain a correction factor, which is found to account for the mismatch in wave forms and which resolves the discrepancy in growth rates. The results illustrate that, when theory is compared with experiments in which the values of certain higher-order (nonlinear) quantities must be deduced from measurements of first-order quantities, great care must be taken to ascertain that the experimental conditions and the theoretical assumptions are indeed compatible to the required order.

1. Introduction

An analysis was presented by Benjamin & Feir (1967) which showed that a weakly nonlinear, uniform, deep-water wave train (a Stokes wave train) is unstable to modulational perturbations of the envelope (corresponding to a pair of side bands around the primary component in the power spectrum). In a following paper by Benjamin (1967), some experimental evidence on the instability was published and the theory was qualitatively verified. However, quantitative comparison between theory and experiment indicated that the theory overpredicted the exponential growth rate by roughly a factor of two. In the discussion, Benjamin suggested that the neglect of dissipative effects in the theory may account for up to one-third of the difference, but the bulk of the discrepancy was left unexplained.

Since then much theoretical effort has been devoted to the search for a uniformly valid equation which can describe the evolution of a nonlinear wave train. The nonlinear Schrödinger equation, first derived by Zakharov (1968) and found to predict well the long-time behaviour of nonlinear wave packets (Yuen & Lake 1975), appears to be the appropriate choice. In fact, it was shown that the results of Benjamin & Feir (1967) are in quantitative agreement with those obtained from a stability analysis of its uniform solution. However, before we can apply the equation to obtain any quantitative information pertaining to the long-time evolution of the nonlinear wave train, it is necessary that the discrepancy between experimental and theoretical initial growth rates be understood and resolved.

The purpose of this note is to demonstrate that the discrepancy is not a result of errors in the analysis or experiment, but rather is caused by a subtle mismatch of theoretical and experimental conditions that may arise when a second-order quantity (such as the parameter ka when used as a measure of the degree of nonlinearity in this system) must be deduced from measurements of first-order quantities (such as the wave amplitude and frequency).

2. Theory and experiment

Consider a uniform, steady, weakly nonlinear, deep-water wave train with amplitude a, wavenumber k and frequency ω (in radians). Correct to $O(k^2a^2)$, the free-surface elevation $\eta(x,t)$ can be expressed as an expansion in the first two harmonics:

$$\eta(x,t) = a\cos(kx-\omega t) + a_2 \cos 2(kx-\omega t), \tag{1}$$

where a_2 is the amplitude of the second harmonic. Stokes (1847) found that one of the conditions for such a steady wave to exist is for a_2 to satisfy

$$a_2 = \tfrac{1}{2}ka^2. \tag{2}$$

Benjamin & Feir (1967) examined the stability of the wave form given by (1) to side-band disturbances of the form

$$\epsilon(x,t) = \epsilon_+ e^{\Omega t}\cos[k(1+\kappa)x-\omega(1+\delta)t] + \epsilon_- e^{\Omega t}\cos[k(1-\kappa)x-\omega(1-\delta)t], \tag{3}$$

where κ and δ are small perturbations in the wavenumber and frequency. The perturbed free-surface elevation $\eta(x,t)+\epsilon(x,t)$ then corresponds to a nearly uniform wave train with a weak amplitude modulation. Benjamin & Feir found that, for a given value of the wave steepness ka of the initial wave train, there exists a range of frequencies centred around the primary frequency ω for which Ω is real and positive, so that the disturbances grow exponentially with time. More precisely, they found that

$$\Omega = \tfrac{1}{2}\delta(2k^2a^2-\delta^2)^{\frac{1}{2}}\omega, \tag{4}$$

so that Ω is real whenever $0 < \delta < 2^{\frac{1}{2}}ka$, and the growth rate is maximum for a given ka when $\delta = \delta_{\max} = ka$.

It should be pointed out at this stage, however, that in obtaining (4) the Stokes condition (2) on a_2 is assumed to be satisfied.

These results can be applied to an experimental situation in which a nearly uniform, weakly nonlinear wave is generated mechanically by a wave paddle at one end of a tank and allowed to propagate down the tank, with measurements of the wave characteristics made at various fixed locations, provided that we transform the theoretical co-ordinates (x,t) into the laboratory co-ordinates $(\bar{x},\bar{t})$ by the transformation

$$(x,t) = (C_g\bar{t}, \bar{x}/C_g), \tag{5}$$

where $C_g = \omega/2k$ is the leading-order group velocity. The temporal instability can then be transformed to a spatial one, and the corresponding spatial growth rate (in $\bar{x}$) becomes

$$\Omega_{\bar{x}} = \delta(2k^2a^2-\delta^2)^{\frac{1}{2}}k. \tag{6}$$

This transformation, which was used by Benjamin & Feir (1967), Benjamin (1967) and Chu & Mei (1970, 1971) for comparing theory with experiment, will be used by us throughout this note. It has been checked that the leading-order value of $\omega/2k$ for the group velocity is sufficiently accurate for the present purposes.

We have performed experiments in a $3 \times 3 \times 40$ ft water-wave tank with a programmable wave maker at one end and a wave-absorbing beach at the other. The wave maker has a frequency response of 1–5 Hz and an amplitude response of 0·01–2 in. The measurements were made with capacitance wave gauges at six locations in the tank: at 5 ft, 10 ft, 15 ft, 20 ft, 25 ft and 30 ft from the wave paddle. The sensitivity of the wave gauges was typically 3 V/in. over a 2 in. range.

The growth rate of the unstable side bands was deduced from the ratio of the energy contained in each side band to that in the primary as obtained from power spectra of measurements made at each tank location. The taking of the ratio removes the first-order effect of dissipation, since the dissipation rates for the side bands and the primary should be nearly identical owing to the small frequency separations between them. There is also a second-order effect caused by dissipation: the weakening of the overall nonlinearity of the system as a result of energy depletion. However, it is expected that for the range of initial side-band growth in which we are interested this second-order effect can be safely neglected.

The three parameters governing the experiments are δ, ka and $\epsilon_{\pm}$. Among these, $\epsilon_{\pm}$ is of least importance, for its value does not affect the growth rate. Its only importance is to distinguish whether we are studying the case of natural side bands ($\epsilon_{\pm} \simeq 0$ except for background noise) or experimentally imposed side bands ($\epsilon_{\pm} > 0$, usually of the order of 1 % of a in our experiments). The remaining two parameters define two types of experiment:

(I) Fix ka, vary δ (including the case $\epsilon_{\pm} \simeq 0$).

(II) Fix δ, vary ka ($\epsilon_{\pm} > 0$).

It should be noted that under condition I the growth rate attains a maximum at $\delta = \delta_{\max}$ and becomes zero when $\delta > 2^{\frac{1}{2}}ka$, while under condition II the growth rate starts to become non-zero for $ka > \delta/2^{\frac{1}{2}}$ and increases monotonically as ka increases.

To compare quantitatively the experimental results with the theory, one must relate the parameters δ and ka to experimentally measured quantities. The value of δ can be simply obtained with accuracy from the separation of the side bands from the primary in the power spectrum, and verified by comparison with the inverse of the modulational time period obtained from the measured wave forms. In cases of natural side-band growth ($\epsilon_{\pm} \simeq 0$), only one pair of side bands, the most unstable pair, emerges from the background noise perturbations to dominate wave-train evolution. In other cases, where imposed side bands are present, either the imposed pair of side bands or the most unstable pair of side bands, or both, can be found depending on the values of $\epsilon_{\pm}$ and δ.

Theoretically, the value of ka should be the steepness of the waves at $t = 0$; or in the laboratory co-ordinates, at $\bar{x} = 0$ (right off the paddle). This is usually obtained by measuring the frequency f (in Hz) and amplitude a at a station reasonably close to the paddle and setting

$$(ka)_{\text{meas}} = (2\pi)^2 g^{-1}(f^2 a)_{\text{meas}}, \tag{7}$$

where g is the acceleration due to gravity. In our case, this initial measurement is taken at 5 ft, a location which is sufficiently far (several wavelengths) from the paddle that transient disturbances associated with the paddle motion can be considered negligible, yet is close enough to the paddle that dissipation and the side-band instability have not had a significant effect. However, by letting ka in the theory take the value $(ka)_{\text{meas}}$ defined by (7) at the 5 ft station (which we denote as $(ka)_{5\,\text{ft}}$), we find that all our experimental data agree with those of Benjamin (1967), and hence suffer from the same discrepancy when compared with theory.

As we have noted earlier, the validity of the theoretical expressions for the growth rate given in (4) and (6) relies heavily on the assumption that the waves under consideration are true nonlinear Stokes waves satisfying (2). This assumption, however, was found to be invalid for our paddle-generated waves. The reason for this is that the wave paddle generates each wave by means of a sinusoidal motion of a given frequency and amplitude. Therefore, while the generated waves may possess a large amplitude, they lack the second harmonic necessary to qualify them as true nonlinear Stokes waves. As they leave the wave paddle, these waves presumably tend to adjust to the Stokes profile. Inspection of power spectra of wave amplitude measurements, however, indicates that the magnitudes of a_2 obtained from the spectra are smaller than the magnitudes of $\frac{1}{2}ka^2$ calculated from the measured frequencies and amplitudes, and that the wave forms still do not satisfy the Stokes condition (2) as far down the tank as 10 ft, at which point the Benjamin–Feir instability has often taken effect. The sampled values of the ratio $a_2/\frac{1}{2}ka^2$ show considerable scatter but are consistently less than unity, indicating that the generated waves are not Stokes waves and that they are in some sense 'less nonlinear' than wave-gauge measurements of their frequencies and amplitudes would indicate. The use of $(ka)_{5\,\text{ft}}$ (or ka measured at any location reasonably near the wave paddle) for ka in the theory would therefore lead to an overprediction of nonlinear effects and an overprediction of the side-band growth rates, which are extremely sensitive to ka.

We remark here that it is not practical to search for a proper location for the initial measurement in the hope that the correct value for ka would be obtained, since the waves usually experience the Benjamin & Feir instability before they have adjusted to the Stokes profile. In fact, the value of $(ka)_{\text{meas}}$ obtained from measurements of frequency and amplitude within 10 ft from the wave paddle is remarkably insensitive to changes in probe location. This is to be expected, since the frequency and amplitude are both first-order quantities and should remain relatively unchanged (the amount of energy redistributed to the second and higher harmonics, although important, is small in magnitude). Our problem arises from having to deduce the value of a second-order quantity characterizing the nonlinearity, such as ka (or a_2), from the measurements of first-order quantities, f and a; when the theoretical assumptions regarding wave shape [in this case the Stokes condition (2)] are not satisfied by the experimental conditions, the deduced value is in error even though the measurements are accurate.

What is required is to establish a relationship between the experimental values of $(ka)_{\text{meas}}$ and the value of ka which correctly characterizes the nonlinearity of the generated wave train. To do this we resort to the theory. Recall that, for cases where no prescribed side bands are present initially ($\epsilon_{\pm} \simeq 0$), only the pair of most unstable side bands dominates wave-train evolution. The theoretically predicted value of δ for this pair of side bands is simply

$$\delta = \delta_{\text{max}} = ka. \tag{8}$$

We now propose that this result holds for our experiments, and that the value of δ_{max} measured can be used to obtain the characteristic ka needed for comparison with theory.

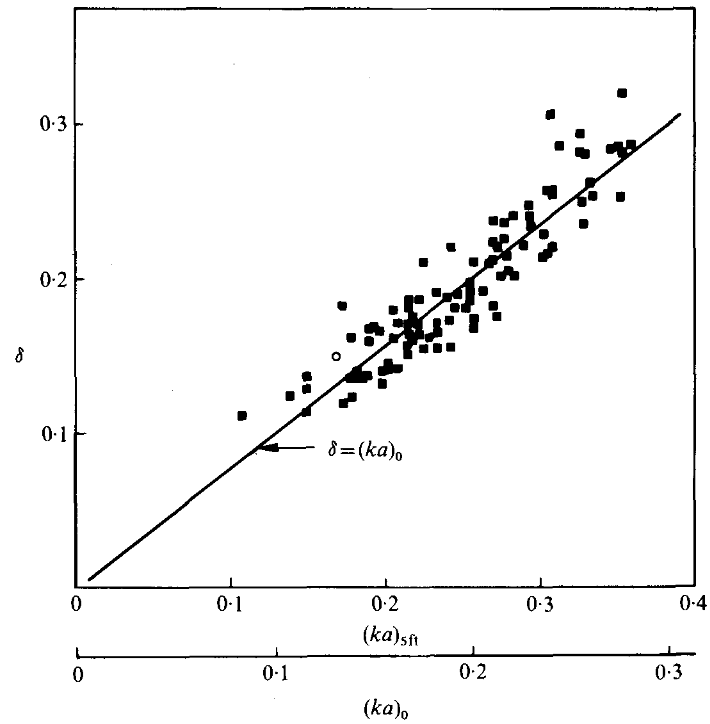

FIGURE 1. Normalized frequency separation δ of dominant side-band components (no imposed initial modulations) as function of the wave steepness $(ka)_{5\,ft}$ measured at $x = 5$ ft and the effective initial wave steepness $(ka)_0$. ■, present results, $2{\cdot}0 \leqslant f_0 \leqslant 3{\cdot}3$ Hz; ○, Benjamin & Feir; ——, least-squares straight-line fit to the data. The double horizontal scale relates $(ka)_{5\,ft}$ and $(ka)_0$; $(ka)_0 = 0{\cdot}78(ka)_{5\,ft}$.

In order that this proposed scheme for determining ka should be correct and useful, the value of ka so obtained should possess the following properties.

(i) It should be somewhat less than $(ka)_{meas}$.

(ii) It should bear a simple relation to $(ka)_{meas}$ at a fixed location.

(iii) It should predict well the growth rate of modulations measured in other, independent experiments, including experiments in which side bands are imposed at frequency separations for which $\delta \neq \delta_{max}$.

(iv) It should lead to good agreement between theory and experiment for the long-time evolution of the wave trains.

A series of experiments to determine δ_{max}, and hence ka, has been performed for a wide range of frequencies and amplitudes. The results are shown in figure 1. One can immediately see that properties (i) and (ii) are well satisfied. A least-squares fit through the data points yields the relation

$$ka(=\delta_{max}) = 0{\cdot}78(ka)_{5\,ft}. \tag{9}$$

When this formula is used in experiments with imposed side bands (of both type I and type II, which includes conditions for which $\delta \neq \delta_{max}$) to convert $(ka)_{meas}$ to ka, comparison of theory and experiment shows quantitative agreement (figure 2), and does not exhibit the discrepancy in the modulation growth rate which would arise if

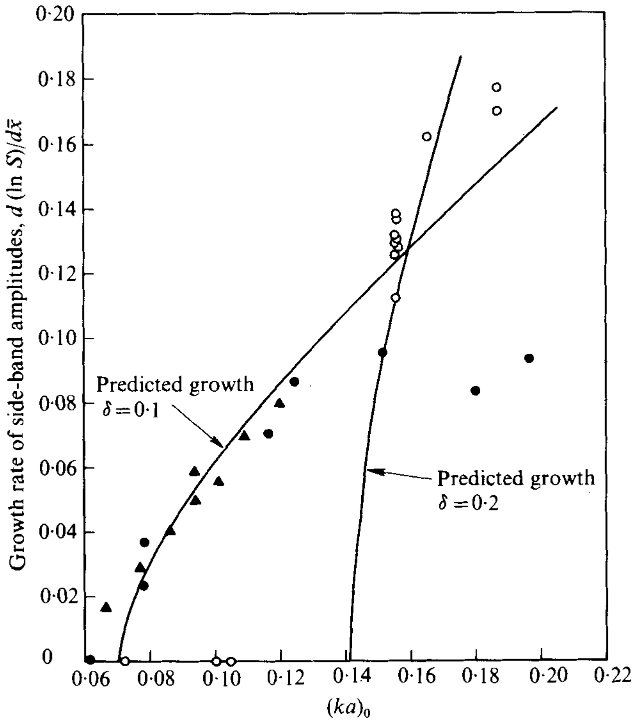

FIGURE 2. Comparison of measured and predicted side-band growth rates $d(\ln S)/d\bar{x}$ for $\delta = 0{\cdot}1$ and $\delta = 0{\cdot}2$ as functions of ka. Here $\bar{x}$ is the normalized propagation distance in units of wavelength: $\bar{x} = kx/2\pi$. Experimental results: ○, $\delta = 0{\cdot}2$, Lake *et al.* (1977); ●, $\delta = 0{\cdot}1$, Lake *et al.* (1977); ▲, Benjamin (1967), referred to our $(ka)_0$.

ka were set equal to $(ka)_{\text{meas}}$. The last property, concerning long-time evolution, has been verified also and is discussed in Lake *et al.* (1977).

The foregoing argument is believed to be sufficiently general to apply to any experimental study of nonlinear waves where the wave paddle undergoes sinusoidal displacements which produce waves having the intended frequency and amplitude but which cannot produce the intended detailed wave shapes. The numerical value of the correction factor, 0·78, may vary somewhat but, in view of the insensitivity of the value $(ka)_{\text{meas}}$ to changes in measurement locations, it is not expected to vary by much. In fact, when we apply this correction to Benjamin's (1967) data, the discrepancy between his data and the predictions of the theory is almost completely removed.†

3. Conclusion

In this note we have shown that, when surface waves of finite amplitude are generated by sinusoidal wave-paddle motions of prescribed amplitude and frequency, the waves that are produced do not have true Stokes wave profiles for they lack suitable

† Benjamin (1967, p. 72) stated that in one measurement he found that $\delta_{\max} = 0{\cdot}15$ while $(ka)_{\text{meas}} = 0{\cdot}17$. This would give a numerical factor of 0·88. Although larger than 0·78, this value appears to lie within the range of scatter of the data shown in figure 1, so that the taking of 0·78 as the best fit is not inconsistent.

second-harmonic components. Such waves are therefore 'less nonlinear' in some sense than a measurement of their amplitudes and frequencies would indicate, and a correction factor must be introduced to compensate for the resulting overestimate of the nonlinearity of the waves when experimental results are compared with theoretical predictions. When a proper correction is made, the comparison between experimental data (including those of Benjamin 1967) and the stability analysis of Benjamin & Feir (1967) shows good quantitative agreement. Since the nonlinear Schrödinger equation reproduces the results of Benjamin & Feir for the initial stage of nonlinear wave-train evolution, removal of the discrepancy in the side-band growth rate allows us to go on to investigate the application of the nonlinear Schrödinger equation to the quantitative description of the long-time evolution. The results of such an investigation are reported in Lake *et al.* (1977).

From an experimentalist's point of view, this note may have a more general implication. With the increasing interest in nonlinear phenomena, it is likely that situations similar to those discussed here may arise. It is our hope that this particular case can serve as an example to illustrate that, whenever the values of certain second-order quantities (such as $a_2/a \propto ka$ as a measure of nonlinearity in our case) must be deduced from measurements of first-order quantities (f and a), great care must be taken to ascertain that the experimental conditions and the theoretical assumptions are indeed compatible.

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